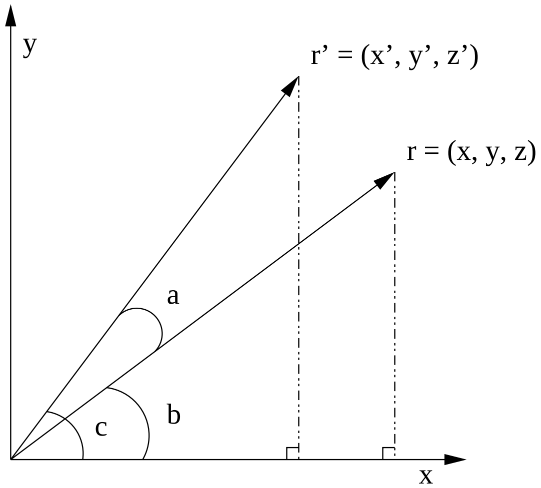
Deriving the rotation matrix

Daniel Martin

November 28, 2005

Let's consider the rotation through an angle, a, of a point **r** $= (x, y, z)$ around the z-axis to **r'** $= (x', y', z')$.



We can express the components of **r**, (x, y, z) in terms of the angle from the x-axis:

$$x = Lcos(b) \tag{1}$$
$$y = Lsin(b) \tag{2}$$
$$z = z$$

where $L = \sqrt{x^2 + y^2}$.

We can do the same for the components of **r'**, (x', y', z'):

$$x' = Lcos(c) \tag{3}$$
$$y' = Lsin(c) \tag{4}$$
$$z' = z$$

We can express c as, $a + b$. Substituting this into equations (3) and (4) we find:

$$
\begin{aligned}
x' &= Lcos(a + b) \\
&= L\left[cos(a)cos(b) - sin(a)sin(b)\right] \qquad (5) \\
y' &= Lsin(a + b) \\
&= L\left[sin(a)cos(b) + cos(a)sin(b)\right] \qquad (6)
\end{aligned}
$$

We can substitute equations (1) and (2) into (5) and (6) to eliminate b and L:

$$
\begin{aligned}
x' &= xcos(a) - ysin(a) \qquad (7) \\
y' &= xsin(a) + ycos(a) \qquad (8) \\
z' &= z
\end{aligned}
$$

We can rewrite these equations in matrix form, **r'** = **R** $\times$ **r** where **R** is the rotation matrix:

$$
\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} cos(a) & -sin(a) & 0 \\ sin(a) & cos(a) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad (9)
$$

Finally let's consider the case where we rotate clockwise by an angle, α, rather than anti-clockwise by an angle, a. ie. $a = -\alpha$:

$$
\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} cos(\alpha) & sin(\alpha) & 0 \\ -sin(\alpha) & cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad (10)
$$

Both equations (9) and (10) are valid the only difference is in how the direction of rotation is defined. If we take the angle of rotation as positive in the clockwise direction then we should use equation (10).